B.Sc. Semester-VI Examination, 2022-23

MATHEMATICS [Honours]

Course ID : 62112 **Course Code : SH/MTH/602/C-14**

Course Title : Ring Theory and Linear Algebra-II

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

Answer **all** the questions.

1. Answer any **five** questions: 2×5=10

a) Let R be a commutative ring with identity. Define polynomial ring $R[x]$ over R.

b) Show that $[1] + [2]x$ is a unit in $\mathbb{Z}_8[x]$.

c) Check irreducibility of the polynomial $5x^4 - 6x^3 + 9x + 6$ over $\mathbb{Q}$ using Eisenstein's Criterion.

d) Use Cayley-Hamilton Theorem to find A^{-1} where $A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$ over $\mathbb{R}$.

e) Let V be an inner product space and $u, w \in V$. If $\langle u, v \rangle = \langle w, v \rangle$ for every $v \in V$ then show that $u = w$.

f) Is $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ diagonalizable over $\mathbb{R}$? Give reasons in support of your answer.

g) Let T and S are two linear operators on $V(F)$. If T and S be invertible then show that TS is invertible.

h) Let V be a finite dimensional inner product space. If T and U are linear operators on V, then prove that $(T + U)^* = T^* + U^*$.

2. Answer any **four** of the following questions: 5×4=20

a) Let R be a PID and p be a non-zero non-unit element of R. Show that p is irreducible if and only if p is prime in R.

b) Show that $\dfrac{\mathbb{Q}[x]}{\langle x^2 - 5 \rangle}$ is a field.

c) Show that a 2×2 real matrix A is diagonalizable iff $(tr(A))^2 > 4|A|$.

d) Let V be a vector space over the field F and W be a subspace of V. Then prove that

$$\dim W + \dim W^0 = \dim V$$

where $W^0 = \{f \mid f \in V^*,\ f(\alpha) = 0\ \forall \alpha \in W\}$ is the annihilator of W.

e) Use Gram-Schmidt process to obtain an orthonormal basis of the subspace of the Euclidean space $\mathbb{R}^3$ with standard inner product generated by the linearly independent set $\{(1, 1, 1), (2, -2, 1), (3, 1, 2)\}$.

f) Let T be a linear operator on a finite dimensional inner product space V. Show that there exists a unique linear operator T^* on V such that $\langle T x, y\rangle = \langle x, T^* y\rangle$ for all $x, y \in V$.

3. Answer any **one** of the following questions:

10×1=10

a) i) Show that $\mathbb{Z}[\sqrt{5}]$ is not a UFD by obtaining an irreducible element there which is not prime.

ii) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y) = (0, x, y)$. Find T^t (transpose of T).

iii) Define normal operator on a finite dimensional inner product space. Let V be a finite dimensional inner product space and T be a normal operator on V. Then prove that α is a characteristic vector for T corresponding to a characteristic value c if and only if α is a characteristic vector for T^* corresponding to the characteristic value $\bar{c}$. 3+3+(1+3)

b) i) Let R be a commutative ring with identity where $R[x]$ is a PID. Prove that R is a field.

ii) Find the Jordan canonical form of the matrix

$$B = \begin{pmatrix} 3 & 2 & -3 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

iii) Let T_1 and T_2 be two self-adjoint linear operators defined on an inner product space $(V, <, >)$. Then prove that the product of T_1 and T_2 is again self-adjoint if and only if T_1, T_2 commute with each other. 4+4+2

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